Usually, most of the properties regarding an abelian cover are encoded in function of the line bundles $\{\mathcal{L}_\chi\}_{\chi \in G^*}$ instead of the $\{D_g\}_{g \in G}$.

For this reason, it is usually challanging to obtain ab. covers with pre-scribed geom. properties. It would be very useful a formula that permits to write the divisors $\{D_g\}_{g \in G}$ in function of the $\{\mathcal{L}_\chi\}_{\chi \in G^*}$
(actually we only have a formula to determine $\{\mathcal{L}_\chi\}_\chi$ in function of the $\{D_g\}_{g \in G^*}$; so we are looking for the opposite problem).

Thm (F. - Pignatelli, 2023)
Let $G = (\mathbb{Z}/p)^k$ a p-elementary abelian group.
Then

$$\boxed{\frac{p^k(p-1)}{2} \sum_{h \in \langle g \rangle} D_h \equiv_{lin} p\Big(\sum_{\chi \notin \ker(g)} \mathcal{L}_\chi - (p-1) \sum_{\chi \in \ker(g)} \mathcal{L}_\chi \Big)}$$

Rem When $p=2$, then $\langle g \rangle = \{0, g\}$, so $\sum_{h \in \langle g \rangle} D_h = D_g$ and we obtain D_g is in function of the $\mathcal{L}_\chi$.

Thm Let us consider a smooth abelian cover $\pi : X \to \mathbb{P}^2$ with group $G = (\mathbb{Z}/p)^k$. Then π is the canonical map of $X \Leftrightarrow$ the building data of π are one of the following:

$p \in \{2,3,5\}$, $k \in \{1,2,3,4\}$, in particular:

$(p,1)$: $D_{\bar{1}}$:= smooth curve of degree $\frac{4p}{p-1}$, $D_{\bar{k}} = 0$ otherwise
$p \in \{2,3,5\}$

$(2,k)$, $k \in \{2,3,4\}$: D_g = smooth curve of deg. $4p^{2-k}$, $g \notin \ker(\varepsilon_1)$;
$D_g = 0$ if $g \in \ker(\varepsilon_1)$

$(3,2)$: $D_{2e_1} = D_{2e_1+e_2} = D_{2e_1+2e_2}$ = smooth curve of deg. 2
$D_g = 0$ otherwise.

proof We denote by $d_g := \deg(D_g)$, $l_\chi := \deg(L_\chi)$ on $\mathbb{P}^2$.
Notice that $l_\chi \geq 1$ $\forall \chi \neq 1$ as X is connected.

Looking at the Lietdke formula

$$H^0(X, K_X) = \bigoplus_{\chi \in G^*} \Big[\prod_{g \in G} t_g^{|g| - r_\chi^g - 1} \Big] \cdot \pi^* H^0(Y, K_Y + L_\chi),$$

then $\pi : X \to \mathbb{P}^2$ is the canonical map of X if and only if there exists only one $\chi \in G^*$ s.t. $h^0(\mathbb{P}^2, L_\chi + K_{\mathbb{P}^2}) = 3$, and $h^0(Y, L_\eta + K_{\mathbb{P}^2}) = 0$ $\forall \eta \in G^*, \eta \neq \chi$.

However, $L_\eta + K_{\mathbb{P}^2} = (l_\eta - 3)H$, so $h^0(\mathbb{P}^2, L_\eta + K_{\mathbb{P}^2}) =$

$$= \begin{cases} \binom{l_\eta - 3 + 2}{2} = \frac{(l_\eta - 1)(l_\eta - 2)}{2} & \text{if } l_\eta - 3 \geq 0 \\ 0 & l_\eta - 3 < 0 \end{cases}$$

This forces $l_\chi = 4$ and $l_\eta \in \{1, 2\}$.

Actually, l_η is forced to be 2 $\forall \eta \neq 1, \chi$

Indeed, $(p-1)\eta + \chi \neq 1, \chi$, so by Pardini

$$\underset{\underset{2}{\parallel}}{l_\eta} + \underset{\underset{2}{\parallel}}{l_{(p-1)\eta + \chi}} = l_\chi + \sum_{g \in G} \left\lfloor \frac{r_\eta^g + r_{(p-1)\eta+\chi}^g}{|g|} \right\rfloor d_g \geq l_\chi = 4$$

$$\Rightarrow \quad l_\eta = l_{(p-1)\eta + \chi} = 2.$$

We have proved $l_\eta = 2 \ \forall \eta \neq 1, \chi$, $l_\chi = 4$.

Now, we use the previous thm. to determine the D_g in terms of L_η:

let $g \in \ker(\chi)$, then:

$$p^{k-1}\frac{(p-1)}{2} \sum_{h \in \langle g \rangle} d_h = \sum_{\eta \notin \ker(g)} l_\eta - (p-1) \sum_{\eta \in \ker(g)} l_\eta$$

all the remaining l_η are equal to 2

here there is $l_0 = 0$ and $l_\chi = 4$

$$= |\ker^c(g)| \cdot 2 - (p-1)\big(2(|\ker(g)| - 2) + 0 + 4\big)$$

$$= 2|\ker^c(g)| + 2|\ker(g)| - 2p|\ker(g)| = 2p^k - 2p \cdot p^{k-1} = 0$$

$\Rightarrow \quad d_h = 0 \quad \forall h \in \ker(\chi).$

Assume now $g \notin \ker(\chi)$. Then

$$\frac{p^{k-1}(p-1)}{2} \sum_{h \in \langle g \rangle} d_h = \sum_{\eta \notin \ker(g)} l_\eta - (p-1) \sum_{\eta \in \ker(g)} l_\eta$$

it contains $l_\chi = 4$ it contains $l_0 = 0$

all the remain ones are $l_\eta = 2$

$$= 4 + 2(|\ker^c(g)| - 1) - (p-1)(2(|\ker(g)| - 1) + 0)$$

$$= 2 + 2|\ker^c(g)| - 2p|\ker(g)| + 2|\ker(g)| + 2(p-1)$$

$$= 2p$$

$\Rightarrow \sum_{h \in \langle g \rangle} d_h = \dfrac{4p}{p^{k-1}(p-1)}$ which is an integer $\Leftrightarrow$

$(p,k) = (2,1), (2,2), (2,3), (2,4), (3,1), (3,2), (5,1), (5,2)$

Let us consider the cases $\boxed{k=1}$:   dual basis of χ ($\chi(g) = e^{\frac{2\pi i}{p}}$)

In this case χ generates $G^* \cong \mathbb{Z}/p$, $G = \langle g \rangle$ and at most one k_g has $d_{k_g} \neq 0$ because X is smooth (and we would have $\langle k_g \rangle \oplus \langle l_g \rangle \to \langle g \rangle$ not inj.)

So $d_{k_g} = \sum_{i=0}^{p-1} d_{ig} = \dfrac{4p}{p-1}$.

We need to check Pardini Formula:

$$p\, l_\chi = \sum_{i=0}^{p-1} i\, d_{ig} = k\, d_{kg} = \frac{4kp}{p-1}$$

$$\Leftrightarrow \quad l_\chi = \frac{4k}{p-1} \quad \text{is an integer} \quad \left(\text{that it is always satisfied}\right)$$

Up to apply an automorphism of $\mathbb{Z}/p$, we can choose $d_{\bar 1} = \frac{4p}{p-1}$ and we would have $l_\varepsilon = 4$, $l_\eta = 2$ $\forall \eta \neq 1, \varepsilon$, $\varepsilon \in (\mathbb{Z}/p)^*$ dual element of $\bar 1$.

Let us consider the cases $\boxed{p=2 \text{ and } k \in \{2,3,4\}}$:

Then $\forall g \in \ker(\chi)$, $d_g = 0$, while

$$\forall g \notin \ker(\chi), \quad \sum_{h \in \langle g \rangle} d_h = \underset{\overset{\shortparallel}{0}}{d_0} + d_g = d_g = \frac{4p}{p^{k-1}(p-1)} = 2^{4-k}$$

Up to apply an automorphism of $(\mathbb{Z}/2)^k$, we can choice $\chi = \varepsilon_1$ dual element of e_1, so we would obtain $d_g = 0$ $\forall g \in \ker(\varepsilon_1)$, and $d_g = 2^{4-k}$ $\forall g \notin \ker(\varepsilon_1)$.

Let us discuss the remaing cases $p \in \{3,5\}$, $k = 2$.

$$d_g = 0 \quad \forall g \in \ker(\chi), \quad \sum_{h \in \langle g \rangle} d_h = \frac{4}{(p-1)} \quad \forall g \notin \ker(\chi).$$

$$\text{Then } \sum_{g \in G} d_g = \sum_{\langle g \rangle \in \mathbb{P}(G)} \underbrace{\left(\sum_{h \in \langle g \rangle} d_h\right)}_{\begin{cases} \frac{4}{p-1} & \text{if } g \notin \ker(\chi) \\ 0 & \text{if } g \in \ker(\chi) \end{cases}} = \frac{4}{p-1} \cdot |\mathbb{P}(G) \setminus \ker(\chi)| = \frac{4}{p-1}\left(\frac{p^2-1}{p-1} - 1\right) = \frac{4}{p-1} \cdot p+1-1 = \frac{4p}{p-1}$$

$$\text{so } \begin{cases} \frac{4}{p-1} & \text{if } \langle g \rangle \neq \ker(\chi) \\ 0 & \text{if } \langle g \rangle = \ker(\chi) \end{cases}$$

Thus, $\sum_{g\in G}(p-1)d_g = 4p = pl_x = \sum_{g\in G} r_x^g d_g \Rightarrow \sum_{g\in G}(p-1-r_x^g)d_g = 0$

so $d_g \neq 0 \iff r_x^g = p-1$.

Up to an automorphism of $(\mathbb{Z}/p)^2$, we can choose $x = \varepsilon_1$. This would imply

$$d_g \neq 0 \iff d_g = \frac{4}{(p-1)} \quad \text{and} \quad r_{\varepsilon_1}^g = p-1.$$

Thus, for $p=3$, then $d_{2e_1} = d_{2e_1+e_2} = d_{2e_1+2e_2} = 2$, $d_g = 0$ otherwise;

It is easy to verify that such building data satisfies Pardini Equations, and so define an abelian covering with group $(\mathbb{Z}/3)^2$.

Instead, for $p=5$, we would have

$$d_{4e_1} = d_{4e_1+e_2} = d_{4e_1+2e_2} = d_{4e_1+3e_2} = d_{4e_1+4e_2} = 1$$

and they define an abelian cover with group $(\mathbb{Z}/5)^2$. However, such ab. cover has $p_g(S) > 3$:

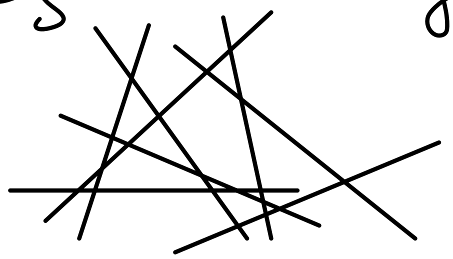
$$5\,l_{2\varepsilon_1} = 3(d_{4e_1} + \cdots + d_{4e_1+4e_2}) = 3\cdot 5 \Rightarrow l_{2\varepsilon_1} = 3.$$

Thus, the cover does not coincide with its canonical map. The case $(p,k) = (5,2)$ must be discarded. ∎

Remark The above theorem has been proved by Du and Gao in 2014 (using another strategy). In F. - Pignatelli, 2023, we generalized this result and classified all smooth $\mathbb{Z}_{/2}^{k}$-double covers $\pi: X \to \mathbb{P}^2$ with $p_g(X)=3$. In another work, I classified all smooth $(\mathbb{Z}/p)^k$-covers of $\mathbb{P}^2$ with $p_g(X)=3$.

Exercise: Let us consider one of the ab. covering of the previous thm: Persson Example:

$G=(\mathbb{Z}/2)^4$, D_g = line $\forall g \notin \ker(\varepsilon_1)$ (We have in total 8 lines in gen. posit.)



We already know that $l_{\varepsilon_1}=4$, $l_\eta=2$ $\forall \eta \neq 1, \varepsilon_1$.

Thus,

$$p_g(X)=h^0(K_{\mathbb{P}^2})+\sum_{\chi\in G^*\setminus 1} h^0(\mathbb{P}^2, K_{\mathbb{P}^2}+L_\chi)$$

$$= h^0(\mathbb{P}^2, \underbrace{K_{\mathbb{P}^2}+L_{\varepsilon_1}}_{(-3+4)H}) = 3 \checkmark$$

$$q(X)=\sum_{\chi\in G^*} \underbrace{h^1(\mathbb{P}^2,(l_\chi-3)H)}_{\overset{\shortparallel}{0}\ \text{always zero on } \mathbb{P}^2} = 0 \checkmark$$

$$\chi(\mathcal{O}_X) = \underset{\substack{\| \\ 1}}{\chi(\mathbb{P}^2)} \cdot 2^4 + \frac{1}{2} \sum_{x \in G^*11}{}' \underbrace{L_x(L_x + K_{\mathbb{P}^2})}_{\substack{\| \\ l_x(l_x-3) \\ \| \\ 4\cdot(4-3)+(2^4-2)\cdot 2(2-3) \\ \| \\ 4+(-28) \\ \| \\ -24}}$$

$$= 16 - 12 = 4 \ \checkmark$$

$$K_X^2 = 2^4\Big(-3 + \frac{1}{2}\underset{\substack{\| \\ 8}}{\sum{}' d_g}\Big)^2 = 2^4 \cdot 1 = 16$$

$$e(X) = 2^4\Big(3 - 2\cdot\big(1-\tfrac{1}{2}\big)\cdot 8 + \big(1-\tfrac{1}{2}\big)^2 \cdot \#\begin{smallmatrix}\text{int. points} \\ 8 \text{ lines}\end{smallmatrix}\Big)$$

$$= 2^4\Big(3 - 8 + \frac{1}{\cancel{4}} \cdot \overset{7}{\cancel{28}}\Big) = 2^4 \cdot 2 = 32$$

By the previous thm, then $\pi: X \to \mathbb{P}^2$ is the can. map of X, so it has deg

$$\deg(\pi) = |G| = 16.$$

This has been for ≈ 35 years the example with the highest degree of the canonical map!!

§13. Equations of an Abelian Cover of $\mathbb{P}^n$ in suitable weighted projective spaces

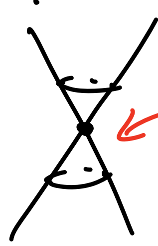
First of all we need some preliminaries on weighted projective spaces and cyclic quotient singularities.

Def Given a normal variety X, a cyclic quotient singularity of type $\frac{1}{r}(a_1, \dots a_n)$ is a point $p \in X$ s.t. there is a local open neigh. $\mathcal{U}$ of p for which $\mathcal{U} \cong \mathbb{C}^n/(\mathbb{Z}/r)$, where $\mathbb{Z}/r$ acts on $\mathbb{C}^n$ as

$$\Phi: \mathbb{C}^n \longrightarrow \mathbb{C}^n$$
$$(z_1, \dots z_n) \longmapsto \left(e^{\frac{2\pi i}{r} a_1} z_1, \dots, e^{\frac{2\pi i}{r} a_n} \cdot z_n\right)$$

Example $Z(X^2 - YZ) \subseteq \mathbb{C}^3$ has a singularity $\frac{1}{2}(1,1)$ at the origin (indeed $Z(X^2 - YZ) \cong \mathbb{C}^2/(\mathbb{Z}/2)$).

A singularity of type $\frac{1}{2}(1,1)$ is also called a node



how it looks a node.

Example 2 (not always something singular)

Consider $\mathbb{C}^2/(\mathbb{Z}/3)$ where $T: \mathbb{C}^2 \to \mathbb{C}^2$
$(x,y) \mapsto (x, e^{\frac{2\pi i}{3}} y)$

then $\mathbb{C}[x,y]^{\mathbb{Z}/3} = \mathbb{C}[x,y^3]$, which is the algebra of $\mathbb{C}^2 \Rightarrow \mathbb{C}^2/(\mathbb{Z}/3) \cong \mathbb{C}^2$, so the origin is still smooth.

We need to take care how $\mathbb{Z}/r$ is acting on $\mathbb{C}^n$; the kind of action (depending on the weights $a_1, \dots a_n$) will determine the kind of singularity (or they may give a smooth point).

We are interested to understand if different weights $a_1, \dots a_n$ may give rise to the same kind of singularities of $\mathbb{C}^n/(\mathbb{Z}/r)$.

Lemma: Let $p \in X$ be an isolated quotient singularity of type $\frac{1}{r}(a_1, \dots a_n)$. Then it is equivalent to a singularity of type $\frac{1}{s}(1, b_2, \dots b_n)$ with $\gcd(b_j, s) = 1$ $\forall j = 2, \dots n$.

Example $\frac{1}{5}(2,3)$ is equivalent to $\frac{1}{5}(2\cdot 3, 3\cdot 3) = \frac{1}{5}(1,4)$.

$\frac{1}{4}(2,2)$ is equivalent to $\frac{1}{2}(1,1)$.

There is a natural reason to introduce weighted projective spaces in Algebraic Geometry:

We want to work with the zero locus of polynomials in an ambient space that is compact.

When the polynomials are homogeneous, then we see the zero locus in the projective space.

How can we do with a set of non-homogeneous polynomials defined in an affine space?

Weighted projective spaces solve this problem, but we need to pay a prize, namely to work in an ambiet space that is compact, but singular.

Def Let $(a_0, a_1, .. a_n)$ be a set of positive integers. We define on $\mathbb{C}^{n+1}$ the following natural equivalence relation:

$$(z_0, \dots z_n) \sim (z_0', \dots z_n') \iff \exists \lambda \in \mathbb{C}^* \text{ s.t. }$$

$$(z_0', \dots z_n') = (\lambda^{a_0} z_0, \dots \lambda^{a_n} z_n)$$

$\mathbb{P}(a_0, a_1, \dots a_n) := \mathbb{C}^{n+1}/\sim$ is called weighted proj. space with weights $(a_0, a_1, \dots a_n)$.

Rem: Up to exchange the variables, we can order $a_0, \dots a_n$:

$$a_0 \le a_1 \le \dots \le a_n.$$

Rem 2: Let us assume $d \mid a_1, a_2, \dots a_n$. Then

$$\begin{aligned} \mathbb{P}(a_0, \dots a_n) &\xrightarrow{\sim} \mathbb{P}\left(\frac{a_0}{d}, \dots \frac{a_n}{d}\right) \\ (z_0 : \dots : z_n) &\longmapsto (z_0 : \dots : z_n) \end{aligned}$$

is an isomorphism.

Furthermore, assume $d \mid a_2, \dots a_n$ and $\gcd(a_1, d) = 1$. Then

$$\begin{aligned} \mathbb{P}(a_0, \dots a_n) &\xrightarrow{\sim} \mathbb{P}\left(a_0, \frac{a_2}{d}, \dots \frac{a_n}{d}\right) \\ (z_0 : \dots : z_n) &\longmapsto (z_0^d : z_1 : \dots : z_n) \end{aligned}$$

is an isomorphism.

Recursively, one can prove that $\mathbb{P}(a_0, \dots a_n)$ is isomorphic to $\mathbb{P}(b_0, \dots b_n)$, where $\gcd(b_0, \dots \hat{b_j}, \dots b_n) = 1$ $\forall j = 0, \dots n$.

Example $\mathbb{P}(2,3,6,9) \cong \mathbb{P}(2,1,2,3) \cong \mathbb{P}(1,2,2,3)$

$\mathbb{P}(2,2,2) \cong \mathbb{P}(1,1,1) = \mathbb{P}^2$

$\mathbb{P}(1,2,2) \cong \mathbb{P}(1,1,1) = \mathbb{P}^2$

$\mathbb{P}(a,b) \cong \mathbb{P}(a,1) \cong \mathbb{P}(1,1) = \mathbb{P}^1$ (all w. proj. sp. of dim. 1 are $\mathbb{P}^1$)

Def A w. proj. space $\mathbb{P}(a_0, a_1, \dots a_n)$ is <u>well-formed</u> if $\forall j = 0, \dots n$ $\gcd(a_0, \dots \hat{a_j}, \dots a_n) = 1$.

Rem Any w. proj. space is isomorphic to a well-formed w. proj. space.

Thm Two well-formed weighted proj. spaces $\mathbb{P}(a_0, \dots a_n)$ and $\mathbb{P}(b_0, \dots b_n)$ are isomorphic
$\iff$ $(a_0, \dots a_n) = (b_0, \dots b_n)$, up to exchange the weights.

$\mathbb{P}^n$ has natural charts to $\mathbb{C}^n$; we are interested to construct the natural charts of $\mathbb{P}(a_0, \dots a_n)$ (well-formed):

$$\mathcal{U}_{z_i} := \{ z_i \neq 0 \} \subseteq \mathbb{P}(a_0, \dots a_n)$$

We define $\xi_{a_j} := e^{\frac{2\pi i}{a_j}}$ be the first a_j-rooth of unity.
Given a complex number $z_j \neq 0$, then we can write $z_j = \rho e^{i\vartheta}$ and so the a_j-roots of z_j are

$$\sqrt[a_j]{\rho}\, e^{i\left(\frac{\vartheta}{a_j} + \frac{2\pi}{a_j} k\right)} = \sqrt[a_j]{\rho}\, e^{i\frac{\vartheta}{a_j}} \cdot \xi_{a_j}^{k}, \quad k = 0, \dots, a_j - 1$$

We consider $\mathbb{C}^n \big/ (\mathbb{Z}/a_j)$, where the action is

$$I : \mathbb{C}^n \longrightarrow \mathbb{C}^n$$
$$(x_0, \dots \hat{x}_j, \dots x_n) \longmapsto (\xi_{a_j}^{a_j - a_0} x_0, \dots \hat{x}_j, \dots, \xi_{a_j}^{a_j - a_n} x_n)$$

We define

$$\Phi_{z_J} : U_{z_J} \xrightarrow{\sim} \mathbb{C}^n / (\mathbb{Z}/a_J)$$

$$(z_0 : \cdots : z_n) \longmapsto \left[\frac{z_0}{\sqrt[a_J]{\rho^{a_0}} \cdot e^{i\frac{\theta}{a_J} a_0} \xi_{a_J}^{k a_0}}, \dots, \hat{z}_J, \dots, \frac{z_n}{\sqrt[a_J]{\rho^{a_n}} \, e^{i\frac{\theta}{a_J} a_n} \xi_{a_J}^{k a_n}} \right]$$

$$= \left[k \cdot \left(\frac{z_0}{\sqrt[a_J]{\rho^{a_0}} \, e^{i\frac{\theta}{a_J} \cdot a_0}}, \dots, \frac{z_n}{\sqrt[a_J]{\rho^{a_n}} \, e^{i\frac{\theta}{a_J} \cdot a_n}} \right) \right]$$

point independ by the choice of k.

It is well defined because $(\lambda^{a_0} z_0 : \cdots : \lambda^{a_n} z_n)$ is sent to:

$z_J = \rho \cdot e^{i\theta}$, $\lambda = \eta \cdot e^{i\nu}$, $\quad \lambda^{a_J} z_J = \eta^{a_J} \rho \, e^{i(\theta + a_J \nu)}$, so its a_J-roots are

$$\eta \sqrt[a_J]{\rho} \, e^{i\left(\frac{\theta}{a_J} + \nu\right)} \cdot e^{i \frac{2\pi}{a_J} \cdot k}, \quad k = 0, \dots a_J - 1$$

Thus

$$\frac{\lambda^{a_0} z_0}{\eta^{a_0} \sqrt[a_J]{\rho^{a_0}} e^{i\left(\frac{\theta}{a_J} + \nu\right)} \cdot \xi_{a_J}^{k a_0}} = \frac{\cancel{\lambda^{a_0}} z_0}{\cancel{\lambda^{a_0}} \sqrt[a_J]{\rho^{a_0}} e^{i \frac{\theta}{a_J} a_0} \xi_{a_J}^{k a_0}} \quad \checkmark$$

The inverse is $\Phi_{z_J}^{-1} : \mathbb{C}^n / (\mathbb{Z}/a_J) \xrightarrow{\sim} U_{z_J}$

$$(z_0, \dots \hat{z}_J, \dots z_n) \longmapsto (z_0 : \cdots : 1 : \cdots z_n)$$

Thus, $\mathbb{P}(a_0, \dots, a_n)$ may have singular points that are cyclic quotient singularities.

Example 1) $\mathbb{P}(1,1,2)$, then locally around $p=(0:0:1)$ we have $\mathbb{C}^2/(\mathbb{Z}/2)$, where the action is

$$(x,y) \mapsto (\xi_2^{2-1}x, \xi_2^{2-1}y) = (\xi_2 x, \xi_2 y) \Rightarrow p \text{ is a}$$

cyclic quotient singularity of type $\frac{1}{2}(1,1)$.

2) $\mathbb{P}(1,2,5)$, $p=(0:0:1)$ has a sing. of type $\frac{1}{5}(5-1, 5-2) = \frac{1}{5}(4,3) \sim \frac{1}{5}(1,2)$

Instead, $p=(0:1:0)$ has a sing. of type

$$\frac{1}{2}(2-1, 2-5) = \frac{1}{2}(1,-3) = \frac{1}{2}(1,1).$$

3) $\mathbb{P}(1,1,2,2)$, then locally to $(0:0:1:0)$ the action is $(x,y,z) \mapsto (\xi_2^{2-1}x, \xi_2^{2-1}y, \xi_2^{2-2}z)$

$$= (\xi_2 x, \xi_2 y, z)$$

$\Rightarrow$ $x=y=0$ is fixed by the action of $\mathbb{Z}/2$

$\Rightarrow$ $\mathbb{P}(1,1,2,2)$ is singular on $z_0=z_1=0$.

Thus, an entire line is singular.

Thm Let $\overset{(p_0:\dots:p_n)}{p} \in \mathbb{P}(a_0:\dots:a_n)$ (well-formed). Then p is singular if and only if, given $I := \{i \in \{0,\dots,n\} \mid p_i \neq 0\}$, then

$$\gcd(a_j : j \in I) > 1.$$

Corollary $\mathbb{P}(\underbrace{\overset{z_0}{1}, 1, \dots \overset{z_n}{1}}_{(n+1)\text{-times}}, \overset{t_1}{a_1}, a_2, \dots \overset{t_r}{a_r}) \setminus \{z_0 = \dots = z_n = 0\}$ is smooth; more precisely,

$$V := \mathbb{P}(1, \dots 1, a_1, a_2 \dots a_r) \setminus \{z_0 = \dots = z_n = 0\} \xrightarrow{\pi} \mathbb{P}^n$$
$$(z_0 : \dots z_n : t_1 : \dots : t_r) \longmapsto (z_0 : \dots z_n)$$

is a vector bundle of rank r over $\mathbb{P}^n$, with cocycles

$$g_{ji} = \begin{pmatrix} (z_i/z_j)^{a_1} & & 0 \\ & \ddots & \\ 0 & & (z_i/z_j)^{a_r} \end{pmatrix}.$$

In other words, it is $\mathcal{O}_{\mathbb{P}^n}(a_1) \oplus \dots \oplus \mathcal{O}_{\mathbb{P}^n}(a_r)$.

proof Let $p = (p_0 : \dots : p_n : t_1 : \dots : t_r) \in V$.

Then at least one $p_i \neq 0$, $i = 0, \dots n$, so $i \in I = \{j \mid p_j \neq 0 \text{ or } t_j \neq 0\}$, which forces

$$\gcd(\text{weights} \mid j \in I) = 1$$

$\Rightarrow p$ is a smooth point.

Let us consider the open sets $\pi^{-1}(U_{z_i})$. We have

$$\pi^{-1}(U_{z_i}) \xrightarrow{\varphi_i} U_{z_i} \times \mathbb{C}^r$$
$$(z_0 : \dots : z_n : t_1 : \dots : t_r) \longmapsto \left((z_0 : \dots : z_n), \left(\frac{t_1}{z_i^{a_1}}, \dots \frac{t_r}{z_i^{a_r}}\right)\right)$$
$$(z_0 : \dots : z_n : z_i^{a_1} x_1 : \dots : z_i^{a_r} x_r) \xleftarrow{\varphi_i^{-1}} ((z_0 : \dots z_n), (x_1, \dots x_r))$$

and the transition functions of V are

$$\varphi_j \circ \varphi_i^{-1} : U_{z_i} \cap U_{z_j} \times \mathbb{C}^r \longrightarrow U_{z_i} \cap U_{z_j} \times \mathbb{C}^r$$
$$((z_0 : \dots : z_n), (x_1, \dots x_r)) \longmapsto \left((z_0 : \dots z_n), \left(\frac{z_i^{a_1}}{z_j^{a_1}} x_1, \dots, \frac{z_i^{a_r}}{z_j^{a_r}} x_r\right)\right)$$

so V is a rank r-vector bundle of $\mathbb{P}^n$ with cocycles $g_{ji} = \mathcal{L}_j \circ \mathcal{L}_i^{-1} = \begin{pmatrix} (z_i/z_j)^{a_1} & & 0 \\ & \ddots & \\ 0 & & (z_i/z_j)^{a_r} \end{pmatrix}$.

These cocycles are exactly those of
$$\mathcal{O}_{\mathbb{P}^n}(a_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^n}(a_n) \qquad \blacksquare$$

Thm Let $\pi : X \to \mathbb{P}^n(z_0 : \cdots : z_n)$ be a std. abelian covering of $\mathbb{P}^n$ with Galois group G and building data
$$\{L_\chi\}_{\chi \in G^*}, \{D_g\}_{g \in G}.$$
Let us define
$$l_\chi := \deg(L_\chi), \chi \in G^* \text{ and } d_g := \deg(D_g), g \in G \text{ on } \mathbb{P}^n.$$

Let us write, $D_g = (f_g = 0)$, so f_g is a homog. polynomial of degree d_g defining the divisor D_g in $\mathbb{P}^n$.
Then
$$X = \left\{ (z_0 : \cdots : z_n : (y_\chi : \chi \in G^* \setminus 1)) \;\middle|\; y_\chi y_{\chi'} = y_{\chi\chi'} \Big(\prod_{g \in G} f_g^{\left\lfloor \frac{r_\chi^g + r_{\chi'}^g}{|g|} \right\rfloor} \Big), \chi, \chi' \in G^* \right\}$$
$$\cap$$
$$\mathbb{P}(1, \cdots 1, l_\chi : \chi \in G^* \setminus 1) \setminus \{z_0 = \cdots = z_n = 0\},$$
the action of G on X is
$$(z_0 : \cdots : z_n : (y_\chi : \chi \in G^* \setminus 1)) \xmapsto{g} (z_0 : \cdots : z_n : (\chi(g) y_\chi : \chi \in G^* \setminus 1))$$
while the quotient map is the projection on the first $n+1$ factors: $\pi : X \to \mathbb{P}^n$ is $(z_0 : \cdots : z_n : (l_\chi : \chi \in G^* \setminus 1)) \mapsto (z_0 : \cdots : z_n)$

proof It is sufficient to remind the def. of a standard ab. cover of $\mathbb{P}^n$ using Pardini Existence thm:

we consider $V(\bigoplus_{\chi\in G^*\setminus 1} \mathcal{L}_\chi) \xrightarrow{\pi} \mathbb{P}^n$, and X is given locally given as follows:

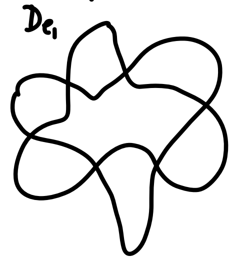
$$X\cap\pi^{-1}(\mathcal{U}_{z_i}) = \left\{ y^i_\chi y^i_{\chi'} = y^i_{\chi\chi'} \left(\prod_{g\in G} (f^i_g)^{\left\lfloor \frac{r^g_\chi + r^g_{\chi'}}{|g|}\right\rfloor}\right)\right\}$$

We observe that by def. $V(\bigoplus_{\chi\in G^*\setminus 1}\mathcal{L}_\chi)$ has cocycles

$$g_{ji} = \begin{pmatrix} (z_i/z_j)^{l_{\chi_1}} & & 0\\ & \ddots & \\ 0 & & (z_i/z_j)^{l_{\chi_{|G|-1}}}\end{pmatrix}$$

so that $V(\bigoplus_{\chi\in G^*\setminus 1}\mathcal{L}_\chi) = \mathbb{P}(1:\dots:1:l_\chi:\chi\in G^*\setminus 1)\setminus\{z_0=\dots=z_n=0\}$ and the thesis follows directly ∎

Example Let us consider the previous $(\mathbb{Z}/2)^2$-cover of $\mathbb{P}^2$: $D_{e_1} = D_{e_1+e_2}$ = smooth quartic of $\mathbb{P}^2$



Then in this case we already computed that $l_{\varepsilon_1}=4$, $l_0=0$, $l_\chi=2\ \forall\chi\neq 0,\varepsilon_1$

The equations are $y_{\varepsilon_1+\varepsilon_2}y_{\varepsilon_2} = y_{\varepsilon_1}$ (so y_{ε_1} is redundant)

$y_{\varepsilon_2}^2 = f_{e_1+e_2}$, $y^2_{\varepsilon_1+\varepsilon_2} = f_{e_1}$ $\Rightarrow$

$$X = \{y^2_{\varepsilon_2} = f_{e_1+e_2},\ y^2_{\varepsilon_1+\varepsilon_2} = f_{e_1}\} \subseteq \mathbb{P}(1,1,1,\overset{l_{\varepsilon_2}}{\overset{\downarrow}{2}},\overset{l_{\varepsilon_1+\varepsilon_2}}{\overset{\downarrow}{2}})\setminus\{z_0=z_1=z_2=0\}.$$